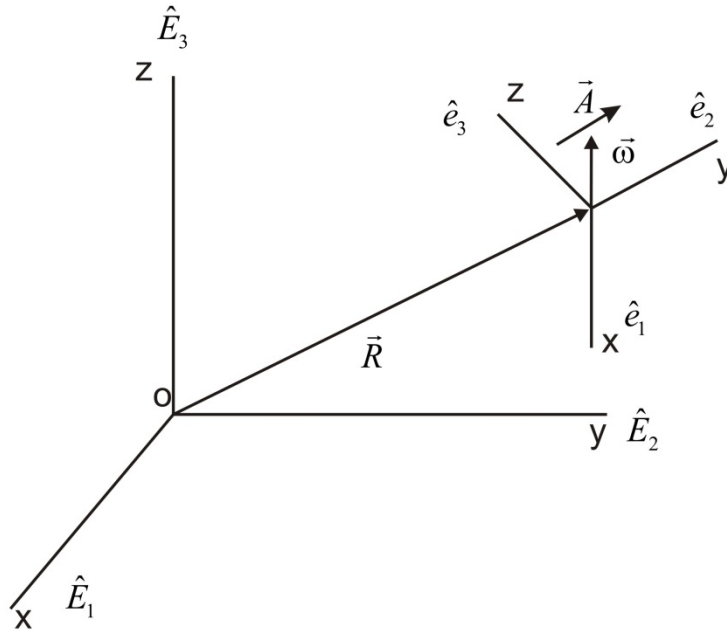


Lecture 3

Kinematics of a particle

Finding derivative of a vector fixed in a moving reference frame.



Let us assume that \vec{A} is a vector fixed in the reference frame xyz . From the above figure we can write

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

$$\Rightarrow \frac{d\vec{A}}{dt} \Big|_{XYZ} = \frac{dA_1}{dt} \Big|_{XYZ} \hat{e}_1 + \frac{dA_2}{dt} \Big|_{XYZ} \hat{e}_2 + \frac{dA_3}{dt} \Big|_{XYZ} \hat{e}_3 + A_1 \frac{d\hat{e}_1}{dt} \Big|_{XYZ} + A_2 \frac{d\hat{e}_2}{dt} \Big|_{XYZ} + A_3 \frac{d\hat{e}_3}{dt} \Big|_{XYZ}$$

where A_1, A_2, A_3 are the body components.

Case (a)

If A_i is fixed in (xyz) reference frame then

$$\frac{dA_i}{dt} \Big|_{XYZ} = 0 = \frac{dA_i}{dt} \Big|_{xyz}$$

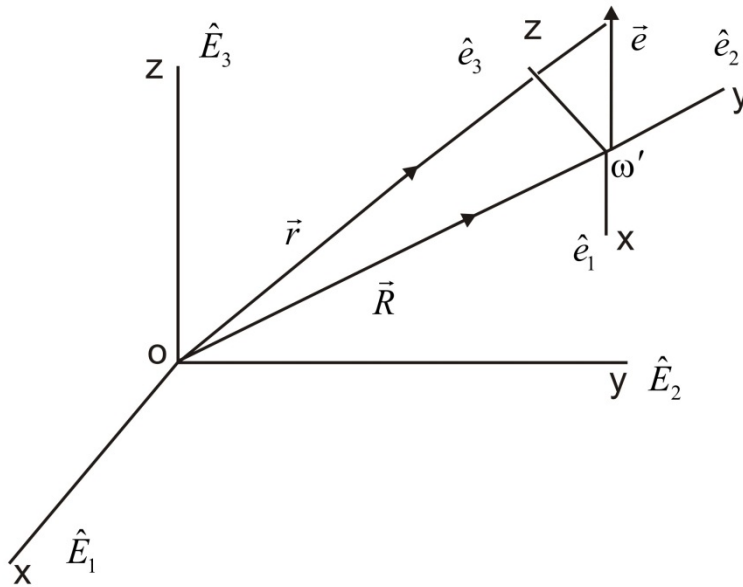
$$\frac{d\vec{A}}{dt} \Big|_{XYZ} = \sum A_i (\vec{\omega} \times \hat{e}_i) = \vec{\omega} \times \vec{A}$$

Case (b)

If A_i are changing in body reference frame then

$$\frac{dA_i}{dt} \Big|_{XYZ} = \frac{dA_i}{dt} \Big|_{xyz}$$

$$\Rightarrow \frac{d\vec{A}}{dt} \Big|_{XYZ} = \frac{d\vec{A}}{dt} \Big|_{xyz} + \vec{\omega} \times \vec{A}$$

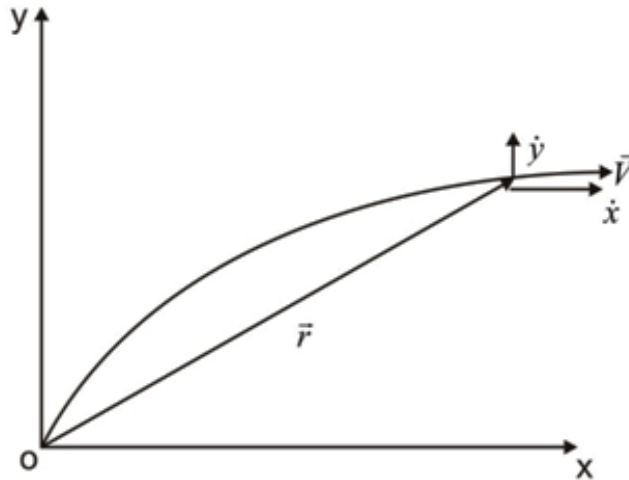


$$\begin{aligned}
\Rightarrow \vec{r} &= \vec{R} + \vec{e} \\
\Rightarrow \frac{d\vec{r}}{dt} \Big|_{XYZ} &= \frac{d\vec{R}}{dt} \Big|_{XYZ} + \frac{d\vec{e}}{dt} \Big|_{XYZ} \\
\Rightarrow \dot{\vec{r}}_{XYZ} &= \dot{\vec{R}}_{XYZ} + \dot{\vec{e}} \Big|_{XYZ} \\
&= \dot{\vec{R}}_{XYZ} + \dot{\vec{e}}_{xyz} + \vec{\omega} \times \vec{e} \\
\frac{d\dot{\vec{r}}}{dt} \Big|_{XYZ} &= \ddot{\vec{r}} = \ddot{\vec{R}}_{XYZ} + \frac{d}{dt} (\dot{\vec{e}}_{xyz}) \Big|_{XYZ} + \frac{d\vec{\omega}}{dt} \times \vec{e} + \vec{\omega} \times \frac{d\vec{e}}{dt} \Big|_{XYZ} \\
&= \ddot{\vec{R}}_{XYZ} + \frac{d}{dt} [\sum \dot{e}_i \hat{e}_i]_{XYZ} + \dot{\vec{\omega}} \times \vec{e} + \vec{\omega} \times (\dot{\vec{e}}_{xyz} + \vec{\omega} \times \vec{e}) \\
&= \ddot{\vec{R}}_{XYZ} + \sum \frac{d\dot{e}_i}{dt} \hat{e}_i \Big|_{XYZ} + \sum \dot{e}_i \frac{d\hat{e}_i}{dt} \Big|_{XYZ} + \vec{\omega} \times \dot{\vec{e}} + \vec{\omega} \times (\vec{\omega} \times \vec{e}) \\
&= \ddot{\vec{R}}_{XYZ} + \sum \ddot{e}_i e_i \Big|_{xyz} + \vec{\omega} \times \dot{\vec{e}} + \sum \dot{e}_i \vec{\omega} \times \hat{e}_i + \dot{\vec{\omega}} \times \vec{e} + \vec{\omega} \times \dot{\vec{e}} + \vec{\omega} \times (\vec{\omega} \times \vec{e}) \\
&= \ddot{\vec{R}}_{XYZ} + \vec{a}_{xyz} + \vec{\omega} \times \sum \dot{e}_i \hat{e}_i + \dot{\vec{\omega}} \times \vec{e} + \vec{\omega} \times \dot{\vec{e}} + \vec{\omega} \times (\vec{\omega} \times \vec{e}) \\
\vec{a}_{xyz} &= \ddot{\vec{R}}_{XYZ} + \vec{a}_{xyz} + \vec{\omega} \times \dot{\vec{e}} + \dot{\vec{\omega}} \times \vec{e} + \vec{\omega} \times \dot{\vec{e}} + \vec{\omega} \times (\vec{\omega} \times \vec{e}) \\
&= \ddot{\vec{R}}_{XYZ} + \vec{a}_{xyz} + 2(\vec{\omega} \times \dot{\vec{e}}) + \dot{\vec{\omega}} \times \vec{e} + \vec{\omega} \times (\vec{\omega} \times \vec{e})
\end{aligned}$$

Projectile motion:

Projectile trajectory in the constant gravitational field is a parabola while in an inverse square gravitational field it is an ellipse. The elliptical trajectory will be proved later. Here, the proof for parabolic trajectory is given.

Assumption: For short distances the earth can be taken to be non-rotating.



$$\ddot{x} = 0 \quad \text{-----(1)}$$

$$\ddot{y} = -g_0 \quad \text{-----(2)}$$

Integrating equation (1)

$$\dot{x} = v \cos \theta = v_{ox}$$

$$\Rightarrow \frac{dx}{dt} = v_{ox} \Rightarrow \int_{x_0}^x dx = v_{ox} \int_{t_0}^t dt$$

$$\Rightarrow x - x_0 = v_{ox} (t - t_0)$$

$$\Rightarrow x = x_0 + v_{ox} (t - t_0)$$

If at time t_i , $v_{ox} = v_o \cos \theta$.

$$\text{Then } x = x_0 + v_o \cos \theta (t - t_0) \quad (3)$$

Similarly, integrating equation(2)

$$\frac{dy}{dt} \Big|_{v_{yo}}^{v_y} = -g_0 (t - t_0)$$

$$\Rightarrow \frac{dy}{dt} = v_{jo} - g_0 (t - t_0)$$

$$\Rightarrow y - y_0 = v_{yo} (t - t_0) - \frac{1}{2} g_0 (t - t_0)^2$$

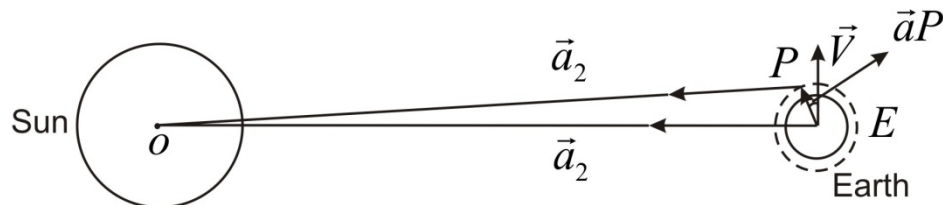
$$= v_o \sin \theta (t - t_0) - \frac{1}{2} g_0 (t - t_0)^2 \quad \text{-----(4)}$$

Eliminating $(t - t_0)$ from equation (3) and (4)

$$\Rightarrow y = y_0 + v_o \sin \theta \cdot \left(\frac{x - x_0}{v_o \cos \theta} \right) - \frac{1}{2} g_0 \left(\frac{x - x_0}{v_o \cos \theta} \right)^2$$

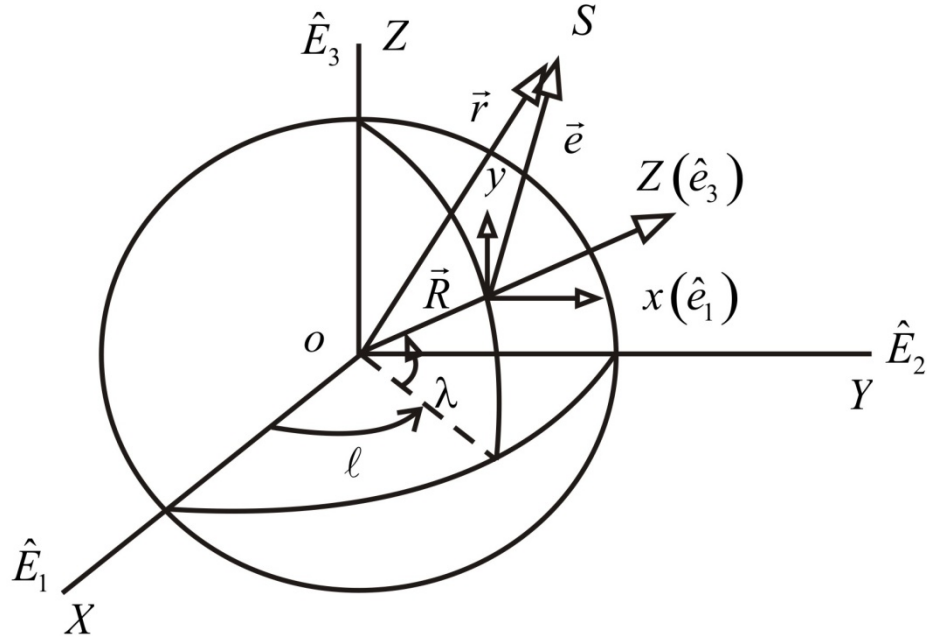
this is an equation of parabola.

Motion relative to the rotating earth



$$\vec{a}_p \approx 0 \text{ as } \vec{a}_1 \text{ and } \vec{a}_2 \text{ are nearly same}$$

The earth orbits the sun. Its orbital motion makes its center accelerate toward the center of mass of the sun-earth system, or in a relaxed way we can say it accelerates towards the center of the sun. Thus, a reference frame fixed at the center of the earth, but not rotating along with the earth, makes a non inertial frame as it is accelerating in reality. Only a reference frame translating with a constant velocity or a static reference frame constitutes an inertial frame. However, the description of the motion of an object w. r. t. the earth's surface i.e., for any earth surface bound object the reference frame fixed at the center of the earth constitutes an inertial frame in practice. Because any object on the surface of the earth is also accelerating towards the sun with almost the same acceleration as the earth and hence for practical purpose such reference frame is inertial.



Let the angular velocity of the coordinate system xyz be $\vec{\Omega}$ in the $E_1E_2E_3$ reference frame. Therefore, the acceleration of an object S w.r.t. the $E_1E_2E_3$ reference frame can be written as.

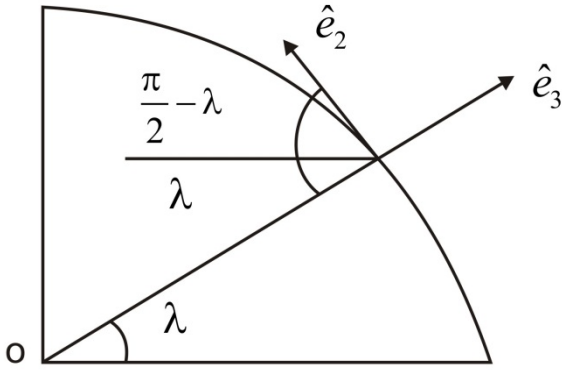
$$\frac{d^2 \vec{r}}{dt^2} = \vec{a} = \left. \frac{d^2 \vec{R}}{dt^2} \right|_{E_1 E_2 E_3} + \left. \frac{d^2 \vec{e}}{dt^2} \right|_{E_1 E_2 E_3}$$

Earlier, we derived the relation

$$\left. \frac{d^2 \vec{r}}{dt^2} \right|_{XYZ} = \ddot{\vec{R}}_{XYZ} + \ddot{\vec{a}}_{xyz} + 2(\vec{\Omega} \times \vec{e}) + \dot{\vec{\Omega}} \times \vec{e} + \vec{\Omega} \times (\vec{\Omega} \times \vec{e})$$

Putting $\dot{\vec{e}} = 0$ $\vec{e} = 0$ in $(\hat{e}_1 \hat{e}_2 \hat{e}_3$ or xyz) reference [\vec{e} is a fixed vector and if $\vec{e} = 0$ then point S lies on the surface of the earth].

$$\begin{aligned} \ddot{\vec{R}}_{E_1 E_2 E_3} &= \ddot{\vec{R}}_{XYZ} = 0 & \vec{a}_{xyz} &= \vec{a}_{e_1 e_2 e_3} = 0 \\ \Rightarrow \frac{d^2 \vec{r}}{dt^2} &= \vec{\Omega} \times (\vec{\Omega} \times \vec{e}), \text{ here } \vec{e} = -R\hat{e}_3 \end{aligned}$$



Now we write components of $\vec{\Omega}$ in $e_1 e_2 e_3$ frame.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/2 - \lambda) & \sin(\pi/2 - \lambda) \\ 0 & -\sin(\pi/2 - \lambda) & \cos(\pi/2 - \lambda) \end{pmatrix} \begin{pmatrix} \cos(l/2 + \pi/2) & \sin(l + \pi/2) & 0 \\ -\sin(l + \pi/2) & \cos(l + \pi/2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix}$$

where l is the longitude and λ is the latitude.

$$\Rightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \Omega \cos \lambda \\ \Omega \sin \lambda \end{bmatrix}$$

$$\vec{\Omega} = \Omega \cos \lambda \hat{e}_2 + \Omega \sin \lambda \hat{e}_3$$

$$\Rightarrow \vec{a}_{xyz} = \vec{\Omega} \times (\vec{\Omega} \times \vec{e})$$

$$= [\Omega \cos \lambda \hat{e}_2 + \Omega \sin \lambda \hat{e}_3] \times [(\Omega \cos \lambda \hat{e}_2 + \Omega \sin \lambda \hat{e}_3) \times (R \hat{e}_3)]$$

$$= +[\Omega \cos \lambda \hat{e}_2 + \Omega \sin \lambda \hat{e}_3] \times [R \Omega \cos \lambda \hat{e}_1]$$

$$\vec{a}_{xyz} = -\Omega^2 R \cos^2 \lambda \hat{e}_3 + \Omega^2 R \sin \lambda \cos \lambda \hat{e}_2$$

Now we can find the acceleration relative to the inertial frame as.

$$\frac{\vec{F}}{m} = \frac{d^2\vec{r}_p}{dt^2} \Big|_{XYZ} + \ddot{\vec{a}}_{xyz} + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times (\vec{\Omega} \times \vec{e}) + \dot{\vec{\Omega}} \times \vec{e}$$

$$\Rightarrow \left[\ddot{\vec{a}} \right]_{xyz} = \frac{\vec{F}}{m} - \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) - \Omega \times \vec{V} - \vec{\Omega} \times (\vec{\Omega} \times \vec{e})$$

$$\text{here } \dot{\vec{\Omega}} \times \vec{e} = 0, \text{ and } \frac{d^2\vec{r}_p}{dt^2} \Big|_{XYZ} = \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$$

Example

Let

$$\vec{r} = (2 + 3t + 4t^2)\hat{e}_1 + (t + 2t^3)\hat{e}_2 + 7t^4.\hat{e}_3$$

Find the velocity and acceleration at

$$t = 4s.$$

$$\frac{d\vec{r}}{dt} = (3 + 8t)\hat{e}_1 + (1 + 6t^2)\hat{e}_2 + 28t^3\hat{e}_3$$

Putting t=4.

$$\frac{d\vec{r}}{dt} = (3 + 32)\hat{e}_1 + (1 + 96)\hat{e}_2 + 28 \times 64\hat{e}_3$$

$$\frac{d\vec{r}}{dt} = 35\hat{e}_1 + 97\hat{e}_2 + 28 \times 64\hat{e}_3$$

$$\left| \frac{d\vec{r}}{dt} \right| = v = \sqrt{35^2 + 97^2 + (28 \times 64)^2} \quad m/s = 111.47 \text{ m/s}$$

Calculating acceleration

$$\frac{d^2\vec{r}}{dt^2} = 8\hat{e}_1 + 12t\hat{e}_2 + 84t^2\hat{e}_3$$

$$= 8\hat{e}_1 + 48\hat{e}_2 + 84 \times 16.\hat{e}_3 \text{ m/s}^2$$

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt} \vec{e}_t.$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$\left(\frac{ds}{dt}\right) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\dot{r} = \frac{dr}{dt}$$

let

$$\vec{r} = (2+3t)\hat{e}_1 + t\hat{e}_2 + 3t\hat{e}_3.$$

$$\Rightarrow r = \sqrt{(2+3t)^2 + t^2 + (3t)^2}$$

$$2r \frac{dr}{dt} = 2 \cdot (2+3t)3 + 2t + 18t.$$

$$v_r = \frac{dr}{dt} = r = \frac{(2+3t)3 + t + 9t}{r}$$

putting $t = 1$.

$$v_r = \dot{r} = \frac{15+1+9}{\sqrt{35}} = \frac{25}{\sqrt{35}}$$

$$v_\theta = r\dot{\theta} =$$

We cannot find V_θ as we need to represent \vec{r} in terms of θ .

However, indirectly we can find V_θ

$$V_\theta = \sqrt{V^2 - V_r^2} = 111.389 \text{ m/s}$$